15/11/16 Čech cohomology

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def X top. space, $\mathcal{C}$ category
A **presheaf** $\mathcal{F}$ on X with values in $\mathcal{C}$ is a functor
$\mathcal{F} : (\text{open sets of } X)^{opp} \to \mathcal{C}$

objects : $U \subset X$ open
morphisms : $\mathrm{Hom}(U,V) = \begin{cases} \{\cdot\} & \text{if } U \subset V \\ \emptyset & \text{else} \end{cases}$

abelian presheaf : presheaf with values in Ab

Example 1 $\mathcal{C}(U) = \{\text{cont. functions on } U\}$

Def 2 **Čech cohomology of an abelian presheaf**
Let $\mathcal{F}$ abelian presheaf on X
$\mathcal{U}$ = set of open sets, $\mathcal{U} = (U_i)_{i \in I}$, fix total order on I.
$\rightsquigarrow$ define a complex of abelian groups

- $\check{C}^q(\mathcal{U}, \mathcal{F}) = \bigoplus_{i_0 < \dots < i_q} \mathcal{F}(U_{i_0} \cap U_{i_1} \cap \dots \cap U_{i_q})$
- $(\partial \varphi)_{i_0, \dots, i_{q+1}} := \sum (-1)^j \varphi_{i_0 \dots \hat{i_j} \dots i_{q+1}} |_{U_{i_0} \cap \dots \cap U_{i_{q+1}}}$

$\qquad \rotatebox{90}{$\in$}$ $\mathcal{F}(U_{i_0} \cap \dots \cap U_{i_{q+1}})$

We set $\check{H}^q(\mathcal{U}, \mathcal{F}) := \dfrac{\mathrm{Ker}(\partial : \check{C}^q \to \check{C}^{q+1})}{\mathrm{Im}(\partial : \check{C}^{q-1} \to \check{C}^q)}$.

If $\mathcal{U}, \mathcal{V}$ are open coverings of X, then there is a natural map $\check{H}^q(\mathcal{U}, \mathcal{F}) \to \check{H}^q(\mathcal{V}, \mathcal{F})$.
Then set $\check{H}^q(X, \mathcal{F}) := \operatorname*{colim}_{\vec{\mathcal{U}}} \check{H}^q(\mathcal{U}, X)$

Notation $\psi_{i_0 i_1 \cdots i_q}$ is the component of $\psi \in \check{C}^q(\mathcal{U}, \mathcal{F})$ in $\mathcal{F}(U_{i_0} \cap \cdots \cap U_{i_q})$ if $i_0 < i_1 < \cdots < i_q$.

It is convenient to define $\psi_{i_0 i_1 \cdots i_q}$ for any sequence $(i_0, \cdots, i_q) \in I^{q+1}$

- If $|\{i_0 \cdots i_q\}| \leq q$ then $\psi_{i_0 i_1 \cdots i_q} = 0$
- If $|\{i_0 \cdots i_q\}| = q+1$, let σ be the permutation such that $\sigma(i_0) < \cdots < \sigma(i_q)$, set $\psi_{i_0 \cdots i_q} = \operatorname{sgn}(\sigma) \psi_{\sigma(i_0) \cdots \sigma(i_q)}$.

Example 2 X top space, A abelian group.
$\rightsquigarrow A_{(x)}$ sky-scraper presheaf, defined by

$$A_{(x)}(U) = \begin{cases} A & \text{if } x \in U \\ 0 & \text{else} \end{cases}$$

then $\check{H}^q(\mathcal{U}, A_{(x)}) = 0 \quad \forall\, \mathcal{U}$ open covering of X if $q \geq 1$.

Pf Let $j \in I$ s.t. $x \in U_j$

Define $K : \check{C}^{q+1}(\mathcal{U}, \mathcal{F}) \to \check{C}^q(\mathcal{U}, \mathcal{F})$

by $(K\psi)_{i_0 \cdots i_q} = \psi_{j i_0 \cdots i_q}$

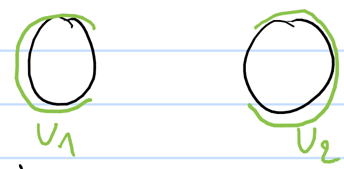
Then we have $(K\partial + \partial K)(\psi)_{i_0 \cdots i_q} = \psi_{i_0 \cdots i_q}$ (computation)

so $K\partial + \partial K = \mathrm{id}$ on $\check{C}^q(\mathcal{U}, \mathcal{F})$
so id is homotopic to 0 and get the claim. ∎

Definition 3 A sheaf $\mathcal{F}$ is a presheaf s.t. for any family $(U_i)_{i\in I}$ of open sets of X, if we are given a family $(\beta_i)_{i\in I}$ of elements $\beta_i \in \mathcal{F}(U_i)$ s.t.

$$\beta_i|_{U_i\cap U_j} = \beta_j|_{U_i\cap U_j} \quad \text{then } \exists!\, \beta\in\mathcal{F}\Big(\bigcup_{i\in I} U_i\Big) \text{ s.t. } \beta_{|U_i} = \beta_i$$

Example 3 $X = S^1$



$\mathcal{U} = (U_1, U_2)$

$\mathcal{F}$ = locally constant sheaf $\mathbb{Z}$

$$\check{C}^0(\mathcal{U},\mathcal{F}) = \mathcal{F}(U_1)\oplus\mathcal{F}(U_2) = \mathbb{Z}\oplus\mathbb{Z}$$
$$\check{C}^1(\mathcal{U},\mathcal{F}) = \mathcal{F}(U_1\cap U_2) = \mathbb{Z}\oplus\mathbb{Z}$$
$$\partial : \check{C}^0 \longrightarrow \check{C}^1$$
$$(a,b) \longmapsto (b-a, b-a)$$
$$\Rightarrow \check{H}^0(\mathcal{U},\mathcal{F}) \simeq \mathbb{Z} \simeq \check{H}^1(\mathcal{U},\mathcal{F})$$

Lemma 1 $\mathcal{F}$ sheaf on X, $\mathcal{U}$ open covering of X. Then $\check{H}^0(\mathcal{U},\mathcal{F}) = \mathcal{F}(X)$.

Sheafified version of Čech cohomology

For $q\geq 0$, set $\mathcal{C}^q(\mathcal{U},\mathcal{F}) := \prod_{i_0<\dots<i_q} i_*\left(\mathcal{F}_{|U_{i_0}\cap\dots\cap U_{i_q}}\right)$

where $i : U\cap U_{i_0}\cap\dots\cap U_{i_q} \longrightarrow U_{i_0}\cap\dots\cap U_{i_q}$

Lemma 2 Let $\mathcal{F}$ be an abelian sheaf on X. The complex $\mathcal{C}^q(\mathcal{U},\mathcal{F})$ is a resolution of $\mathcal{F}$, that is, there is a map $\varepsilon : \mathcal{F}\longrightarrow\mathcal{C}^0(\mathcal{U},\mathcal{F})$ s.t. we get an exact complex
$$0\to\mathcal{F}\to\mathcal{C}^0(\mathcal{U},\mathcal{F})\xrightarrow{\partial}\mathcal{C}^1(\mathcal{U},\mathcal{F})\to\cdots$$

Pf 1) def of ε : $\varepsilon : \mathcal{F}(U) \longrightarrow \prod_{i\in I}\mathcal{F}(U_i\cap U)$
$$\alpha \longmapsto (\alpha_{|U\cap U_i})_{i\in I}$$

2) Exactness at $\mathcal{C}^0(\mathcal{U},\mathcal{F})$ ...

3) Exactness everywhere else

suffices to show on the stalks.

Let $x\in X$. Fix $j\in I$ s.t. $x\in U_j$.

Define $k : \mathcal{C}^q(\mathcal{U},\mathcal{F})_x \longrightarrow \mathcal{C}^{q-1}(\mathcal{U},\mathcal{F})_x$ as follows

$\alpha_x \in \mathcal{C}^q(\mathcal{U},\mathcal{F})_x$

Let $V\subset U_j$ representing α_x : $\alpha_x = [\alpha\in\mathcal{C}^q(\mathcal{U},\mathcal{F})(V)]$.

Set $(k\alpha)_{i_0\dots i_{q-1}} = \alpha_{j\,i_0\dots i_{q-1}}$
$$\mathcal{F}(V\cap U_{i_0}\cap\dots\cap U_{i_{q-1}}) = \mathcal{F}(V\cap U_j\cap U_{i_0}\cap\dots\cap U_{i_{q-1}})$$

Then as in example 2, get the claim. ∎

Theorem 1 X top. space, $\mathcal{U}$ open covering of X.

① $\forall q \geq 0$, $\exists$ application $\check{H}^q(\mathcal{U}, \mathcal{F}) \longrightarrow H^q(X, \mathcal{F})$, functorial in $\mathcal{F}$.

usual sheaf coho defined via derived functors.

② If X is a separated noetherian scheme, $\mathcal{U}$ is an affine open covering of X, $\mathcal{F}$ quasi-coherent on X, then $\forall q \geq 0$, the map in ① is an iso.

NB The map in ① is defined as follows:

$0 \to \mathcal{F} \xrightarrow{\varepsilon} \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$ resolution of $\mathcal{F}$

$0 \to \mathcal{F} \longrightarrow I^\bullet$ injective resolution

Comparing these two resolutions gives a map $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow I^\bullet$

Take global sections + cohomology.