Thomas Gobet Derived functors
22/11/16

Definition A covariant functor F from $Sh(X)$ to Ab is a way to associate to each $\mathcal{F} \in Sh(X)$ an abelian group $F(\mathcal{F})$ and to any map $f : \mathcal{F} \to \mathcal{G}$ in $Sh(X)$ a homomorphism of abelian groups
$F(f) : F(\mathcal{F}) \to F(\mathcal{G})$ s.t.:
- $F(id_{\mathcal{F}}) = id_{F(\mathcal{F})}$ $\forall \mathcal{F} \in Sh(X)$
- $F(\phi \circ \psi) = F(\phi) \circ F(\psi)$ $\forall \phi, \psi$

F is additive if $F(\phi + \psi) = F(\phi) + F(\psi)$ $\forall \phi, \psi \in Hom(\mathcal{F}, \mathcal{G})$
additive F is exact if for exact sequence
$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0 \quad \text{in} \quad Sh(X)$$
the induced sequence $0 \to F(\mathcal{F}) \to F(\mathcal{G}) \to F(\mathcal{H}) \to 0$ is exact
left exact, right exact ...

Example It can be shown that $\Gamma(X, -) : Sh(X) \to Ab$, $\mathcal{F} \mapsto \mathcal{F}(X)$
is left exact.

Definition A sheaf $\mathcal{J} \in Sh(X)$ is injective if the contravariant functor $Hom(-, \mathcal{J})$ is exact.

NB: can reformulate as follows
$$\begin{tikzcd} \forall \mathcal{F} \arrow[r, hook, "\forall"] \arrow[d, "\forall"'] & \forall \mathcal{G} \arrow[dl, dashed, "\exists"] \\ \mathcal{J} & \end{tikzcd}$$

Proposition Let $\mathcal{F} \in Sh(X)$. Then there exists an injective resolution of $\mathcal{F}$ in $Sh(X)$, that is, an exact complex of the form
$$0 \to \mathcal{F} \xrightarrow{\varepsilon} I^0 \xrightarrow{\partial^0} I^1 \to \cdots$$
where I^j is injective

Definition Let $F : Sh(X) \to Ab$ be a left-exact, covariant, additive functor. Let $0 \to \mathcal{F} \xrightarrow{\varepsilon} \mathcal{J}^\bullet$ be an injective resolution of $\mathcal{F}$.
Apply F to this resolution: get a complex
$$0 \to F(\mathcal{F}) \to F(\mathcal{J}^0) \to F(\mathcal{J}^1) \to \cdots$$
Set $R^iF(\mathcal{F}) =$ coho of this complex at i.

Thm 1 For $\mathcal{F} \in Sh(X)$, $R^i F(\mathcal{F})$ is independant on the chosen resolution up to canonical isomorphism.

Defines an additive functor $Sh(X) \to Ab$

We have $R^0 F(\mathcal{F}) = F(\mathcal{F})$.

Pf First, we prove that, for $\mathcal{F} \xrightarrow{f} \mathcal{G}$, and two injective resolutions $\mathcal{I}^\bullet$ and $\mathcal{J}^\bullet$, then f can be extended

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r, "i_{\mathcal{F}}"] \arrow[d, "f"] & \mathcal{I}^0 \arrow[r, "\partial^0"] \arrow[d, "\varphi^0"] & \mathcal{I}^1 \arrow[r] \arrow[d, "\varphi^1"] & \mathcal{I}^2 \arrow[r] & \cdots \\
0 \arrow[r] & \mathcal{G} \arrow[r, "i_{\mathcal{G}}"] & \mathcal{J}^0 \arrow[r] & \mathcal{J}^1 \arrow[r] & \mathcal{J}^2 \arrow[r] & \cdots
\end{tikzcd}$$

Get φ^0 by injectivity of $\mathcal{J}^0$

For φ^1: is 0 so vanishes on $\mathrm{Im} \to \mathcal{I}^0 = \ker \partial^0$

So is defined on $\mathcal{I}^0/_{\ker \partial^0} \simeq \mathrm{Im}\, \partial^0$ vrai dans les catégories abéliennes

$\Rightarrow$ Get φ^1 by injectivity of $\mathcal{J}^1$.

We claim that if $\varphi^\bullet, \psi^\bullet$ are two extensions of f, then they are homotopic.

It means: we can find $h^i : \mathcal{I}^i \to \mathcal{J}^{i-1}$ s.t.

$$\varphi^i - \psi^i = h^{i+1} \partial^i_{\mathcal{I}} + \partial^{i-1}_{\mathcal{J}} h^i$$

We first set $h^0 = 0$. We remark that $(\varphi^0 - \psi^0) \circ i_{\mathcal{F}} = i_{\mathcal{G}} \circ (f - f) = 0$

So $\varphi^0 - \psi^0$ factors through $\mathrm{coker}(i_{\mathcal{F}})$

$$\overline{\varphi^0 - \psi^0} : \mathrm{coker}(i_{\mathcal{F}}) \to \mathcal{J}^0$$

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, "f"] & \mathcal{I}^0 \arrow[r] \arrow[d, "\varphi^0-\psi^0"] & \mathcal{I}^1 \arrow[r] \arrow[dl, "h^1"'] \arrow[d, "\beta \,|\, \varphi^1-\psi^1"] & \mathcal{I}^2 \arrow[dl, "h^2"] \arrow[r] & \cdots \\
0 \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{J}^0 \arrow[r] & \mathcal{J}^1 \arrow[r] & \cdots
\end{tikzcd}$$

h^1 s.t. $\varphi^0 - \psi^0 = h^1 \partial^0_{\mathcal{I}}$

Set $\beta = \varphi^1 - \psi^1 - \partial^0_{\mathcal{J}} h^1$

Want to show: $\exists\, h^2$ s.t. $\beta = h^2 \circ \partial^1_{\mathcal{I}}$

Then go on inductively. $\square$

Now, if $\mathcal{F} \longrightarrow \mathcal{I}^\bullet$ and $\mathcal{F} \to \mathcal{J}^\bullet$ are two resolutions of $\mathcal{F}$ then we have extensions

$$\begin{array}{ccc} \mathcal{F} & \to & \mathcal{I}^\bullet \\ {\scriptstyle id}\downarrow & & \downarrow \varphi^\bullet \\ \mathcal{F} & \to & \mathcal{J}^\bullet \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{F} & \to & \mathcal{J}^\bullet \\ {\scriptstyle id}\downarrow & & \downarrow \psi^\bullet \\ \mathcal{F} & \to & \mathcal{I}^\bullet \end{array}$$

By the unicity, $\varphi^\bullet \circ \psi^\bullet$ is homotopic to $id_{\mathcal{J}^\bullet}$

$\psi^\bullet \circ \varphi^\bullet$ ——————— $id_{\mathcal{I}^\bullet}$

Thus $\varphi^\bullet$ induces an isomorphism of the cohomologies of $F(\mathcal{I}^\bullet)$ and $F(\mathcal{J}^\bullet)$.

$R^0 F(\mathcal{F}) = F(\mathcal{F})$: clear because $\mathcal{F}$ is left exact.

<u>Definition</u> $\mathcal{F} \in Sh(X)$, set $F = \Gamma(X, -)$

The i-th cohomology group of $\mathcal{F}$ is

$$H^i(X, \mathcal{F}) := R^i F(\mathcal{F})$$

Given any short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ of sheaves, there are natural maps $\partial^i : R^i F(\mathcal{F}'') \to R^{i+1} F(\mathcal{F}')$ inducing a long exact sequence $\cdots \longrightarrow R^i F(\mathcal{F}') \to R^i F(\mathcal{F}) \to R^i F(\mathcal{F}'') \to R^{i+1} F(\mathcal{F}') \to \cdots$