1) Recall sheaf cohomology (Lucas)

Let X be a complex variety
If $\mathcal{F}$ is a sheaf on X then $H^k(X, \mathcal{F}) := R^k\Gamma(X, \cdot)(\mathcal{F})$

$H^k_c(X, \mathcal{F}) := R^k \Gamma_c$

global sec^n with cpact support

If $Z \subset X$ closed, $U := X/Z$

then $\exists$ long exact sequence

$$\cdots \longrightarrow H^k_c(U, \mathcal{F}|_U) \longrightarrow H^k_c(X, \mathcal{F}) \longrightarrow H^k_c(Z, \mathcal{F}|_Z)$$
$$\longrightarrow H^{k+1}_c \cdots$$

Notation: $H^k_{(c)}(X) := H_{(c)}(X, \mathbb{Q})$

Fact $H^k_c(\mathbb{C}^d) = \begin{cases} \mathbb{Q} & \text{if } k = 2d \\ 0 & \text{if } k \neq 2d \end{cases}$

Consequence: if $X = X_n \supset X_{n-1} \supset \cdots \supset \cdots$ where each X_j is closed and $X_j \setminus X_{j-1} \simeq \mathbb{C}^{d_j}$

Then $\dim H^k_c(X) = \#$ strata of dim $\frac{k}{2}$.

Moreover if X is projective then $H^k(X) = H^k_c(X)$

For Schubert varieties: $\dim H^k(X_w) = \# \{x \leq w : \ell(x) = \frac{k}{2}\}$

The Poincaré polynomial is ②

$$P_W(t) := \sum_{k\geq 0} \dim H^k(X_w)\, t^k = \sum_{x\leq w} t^{2\ell(x)}$$

2) Intersection Cohomology of Schubert varieties

We have determined $IC^\cdot(X_w)$

$\rightsquigarrow$ Consider $\mathcal{H}^q IC^\cdot(X_w)$ (= sheaf)

Let us Compute

$$H^p(X_w, \mathcal{H}^q IC^\cdot(X_w))$$

$$X_w = \bigcup_{x\leq w} C_x \quad \text{stratific}^{n}$$

We know that $\mathcal{H}^q IC^\cdot(X_w)\Big|_{C_x} \cong \mathbb{Q}^{\oplus a^q_{x,w}}$

$\uparrow$ Coeff. of t^q in $Q_{x,w}(t)$

hence $\dim H^p(C_x, \mathcal{H}^q IC(X_w)) = \begin{cases} a^q_{x,w} & \text{if } p = 2\ell(x) \\ 0 & \text{if } p \neq 2\ell(x) \end{cases}$

Using the long exact sqce:

$$\dim H^p(X_w, \mathcal{H}^q IC^\cdot(X_w)) = \sum_{\substack{x\leq w \\ 2\ell(x)=p}} a^q_{x,w}$$

$\uparrow$

this is not the "intersec^n Cohomology of X_w" $IH^\cdot(X_w)$

$\uparrow$ the hyperCohomology of $IC^\cdot(X_w)$

We know that there is a spectral sqce $E_r^{p,q}$

s.t.

- $E_r^{p,q} \Rightarrow \mathbb{H}^{p+q}(X_w)$
- $E_2^{p,q} = H^p(X_w, \mathcal{H}^q IC^\bullet(X_w))$

In fact this spectral sqce degenerates because

$$H^p(X_w, \mathcal{H}^q IC^\bullet(X_w)) = 0 \text{ if } p \text{ is odd or } q \text{ is odd (ppty of KL polynomials)}$$

hence the differential

$$\begin{array}{lll} E_2^{p,q} \longrightarrow E_2^{p+2,q-1} & \text{is zero} \\ \quad\| & \\ E_3^{p,q} \longrightarrow E_3^{p+3,q-2} & \text{is zero} \\ \quad\| & \\ \quad\vdots & \\ \quad\| & \\ E_r^{p,q} & \end{array}$$

Whence $\dim \mathbb{H}^m(X_w) = \sum_{p+q=m} \dim H^p(X_w, \mathcal{H}^q IC(X_w))$

⇝ We get the Poincaré polynomial

$$\mathbb{P}_w(t) := \sum_m \dim \mathbb{H}^m(X_w)\, t^m$$

$$= \sum_m \sum_{p+q=m} \sum_{\substack{x \le w \\ 2\ell(x) = p}} a_{x,w}^q\, t^p t^q$$

$$= \sum_{x \le w} \underbrace{\left(\sum_q a_{x,w}^q t^q\right)}_{Q_{x,w}(t)} t^{2\ell(x)}$$

$$P_w(t) = \sum_{x \le w} t^{2\ell(x)}$$

$$\mathbb{P}_w(t) = \sum_{x \le w} Q_{x,w}(t)\, t^{2\ell(x)}$$

<u>Rmk</u>: X_w smooth $\Rightarrow$ $Q_{x,w}(t) = 1 \quad \forall x \le w$

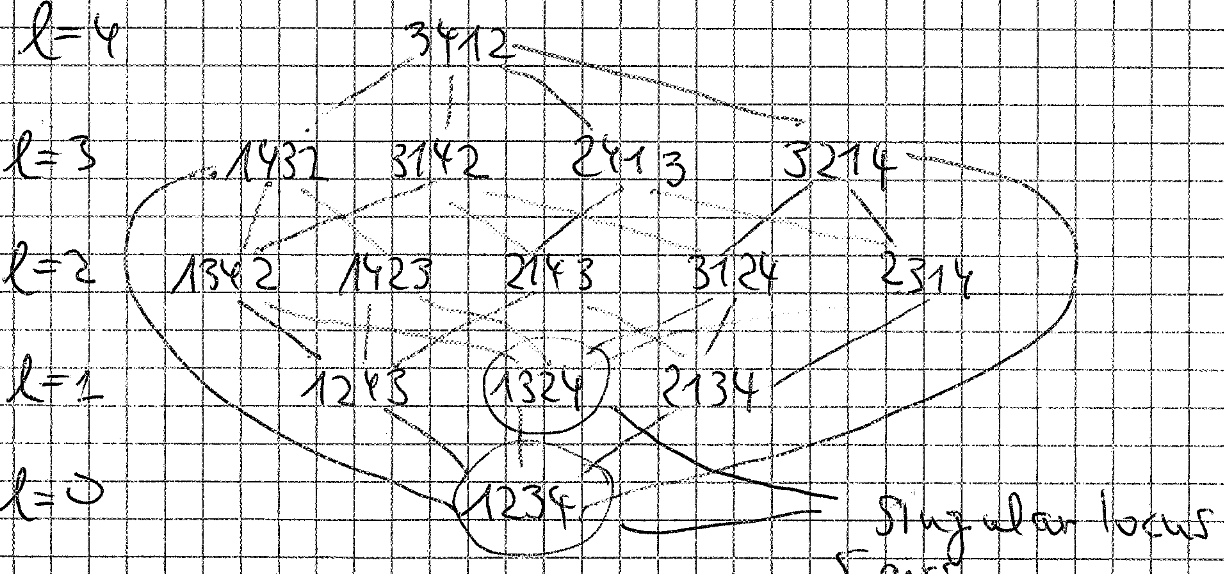
$\Rightarrow$ $P_w(t) = \mathbb{P}_w(t)$ in that case

3) <u>Example</u> in type A_3

let $w = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} = 3412 = s_2 s_1 s_3 s_2$ $\quad \ell = 4$

$\ell(w) = 4 = \dim X_w$

Schubert cells



Thus $P_w(t) = t^8 + 4t^6 + 5t^2 + 3t^2 + 1$

<u>Not</u> palindromic $\Rightarrow$ X_w does not satisfy Poincaré duality $\Rightarrow$ X_w is singular and rationally singular

Bruhat graph of X_w put an edge $u - v$ if $t := u^{-1}v$ is a reflection

We have $C_w = t^{-4}\left(\sum_{\substack{x \leq w \\ x \notin \{1324, 1234\}}} 1 \cdot e_x + (1+t^2)(e_{1324} + e_{1234}) \right)$

$$\Rightarrow \quad Q_{x,w}(t) = \begin{cases} 1 & \text{if } x \notin \{1324, 1234\} \\ 1+t^2 & \text{if } x \in \{1324, 1234\} \end{cases}$$

$P_w(t) = t^8 + 4t^6 + 6t^4 + 4t^2 + 1$

4) Rational smoothness

Def: a variety X is rationally smooth at x if

$$H_\bullet(X, X \setminus \{x\}; \mathbb{Q}) = \mathbb{Q}[\dim X]$$

Lemma (Borho-MacPherson)

$$X \text{ rationally smooth} \iff IC^\bullet(X) \cong \mathbb{Q}[-\dim X]$$

(so smooth $\Rightarrow$ rat. smooth

rat. smooth $\Rightarrow$ $P(t) = P(t)$ $\Rightarrow$ P is palindromic)

For Schubert v'ties we get X_w is rat. smooth

$\iff$ $P_w(t)$ palindromic $\iff$ $Q_{x,w}(t) = 1 \ \forall x \leq w$

There is also

Theorem (Carrell–Peterson) in ~~terms~~ types ADE

X_w smooth $\iff$ X_w is rat. smooth

Example: C_2 $\quad w = s_1 s_2 s_1 \quad$ Bruhat graph

$$\begin{tikzcd}[column sep=small]
\ell=3 & & s_1s_2s_1 \arrow[dl, dash] \arrow[dr, dash] & \\
\ell=2 & s_1s_2 \arrow[d, dash] \arrow[drr, dash] & & s_2s_1 \arrow[d, dash] \arrow[dll, dash] \\
\ell=1 & s_1 \arrow[dr, dash] & & s_2 \arrow[dl, dash] \\
\ell=0 & & 1 &
\end{tikzcd}$$

hence $P_w(t) =$
$IP_w(t) = t^6 + 2t^4 + 2t^2 + 1$

X_w is rat. smooth but singular

$V = \mathbb{C}^4$ is endowed with ω symp. form

with $\omega(e_1, e_4) = 1 = -\omega(e_4, e_1)$

$\omega(e_2, e_3) = 1 = -\omega(e_3, e_2)$

$\omega(e_k, e_\ell) = 0$ otherwise

Let $p := (\underbrace{\langle e_1 \rangle}_{F_1}, \underbrace{\langle e_1, e_2 \rangle}_{F_2})$ isotropic flag since since $F_2 = F_2^{\perp}$

In type C $\quad X = G/B = \{(V_1, V_2)$ isotropic flags ie $V_2 = V_2^{\perp}\}$

$X_w = \{(V_1, V_2) \ \dim V_2 \cap F_2 \geq 1\}$

Affine neighborhood of p in X

$U = \{(\underbrace{\langle e_1 + x e_2 + y e_3 + z e_4 \rangle}_{\varepsilon_1}, \langle \varepsilon_1, e_2 + t e_3 + (xt - y) e_4 \rangle) \ x, y, z, t \in \mathbb{C}\} \simeq \mathbb{C}^4$

$(x,y,z,t) \in U \cap X_w \iff \begin{vmatrix} y & 1 \\ z & xt+y \end{vmatrix} = 0$ 7

at $p=(0,0,0,0)$

the tangent space is $\mathbb{C}^4$ of $\dim > \dim X_w$.