29/11/16 **Intersection cohomology**

From Frances Kirwan : an introduction to intersection homology.

① Introduction

Let X be a smooth complex projective variety and let $H^i(X) = H^i(X, \mathbb{C})$.
It has the following properties :

Hodge decomposition : $H^i(X) = \bigoplus_{p+q=i} H^{p,q}(X)$, and $H^{p,q}(X) = \overline{H^{q,p}(X)}$

$\mathbb{C}$-subspaces

Poincaré duality : There is a non-degenerate pairing
$H^i(X) \otimes H^{2n-i}(X) \longrightarrow \mathbb{C}$
$\alpha \otimes \beta \mapsto \alpha \cup \beta \in H^{2n}(X) = \mathbb{C}$

Lefschetz hyperplane theorem : Let $H \subset \mathbb{P}^m$ be a generic hyperplane.

Then res : $H^i(X) \longrightarrow H^i(X \cap H)$ is an iso if $i < n-1$
and is injective if $i = n-1$

Hard Lefschetz : $\cup [H]^i : H^{n-i}(X) \longrightarrow H^{n+i}$ is an isomorphism

Hodge index theorem : Let $\eta \in H^{p,q}_{prim}(X) = \{\eta \in H^{p,q}(X) : [L]^{n-p-q+1} \cup \eta = 0\}$
Then $\langle i^{p-q} (-1)^{\frac{(n-p-q)(n-p-q-1)}{2}} [L]^{n-p-q} \cup \bar{\eta}, \eta \rangle \in \mathbb{R}_{>0}$.

Those results fail for singular projective varieties.

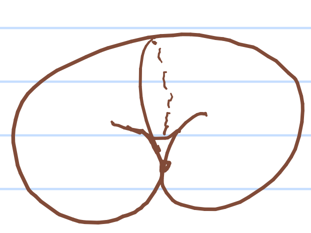
Example 1 $X = \{yz = 0\} \subset \mathbb{P}^2$

$=$ 

Then $H^0(X) = \mathbb{C} \oplus \mathbb{C}$
$H^1(X) = 0$
$H^2(X) = \mathbb{C}$

contradict $\begin{cases} \text{Poincaré duality} \\ \text{Hard Lefschetz} \end{cases}$

Example 2 $X = \{x^3 + y^3 + xyz = 0\}$ elliptic curve with singular point $[0:0:1]$



Sphere with 2 pts identified

$H^0(X) = \mathbb{C}$
$H^1(X) = \mathbb{C}$ $\longrightarrow$ contradicting Hodge decomposition.
$H^2(X) = \mathbb{C}$

L_2 - cohomology

Let $X \subset \mathbb{P}^m$ be singular , $\Sigma = X^{sing}$.
The Fubini - Study metric on $\mathbb{P}^m$ restricts to a Riemannian metric on $X \setminus \Sigma$ $\rightsquigarrow$ metric on each $\Lambda^i T^*(X \setminus \Sigma)$
Let $L^i(X \setminus \Sigma) \subset A^i(X \setminus \Sigma)$ be the space of square-integrable differential i-forms on $X \setminus \Sigma$
Define $H^i_{(2)}(X) = \dfrac{\ker(L^i(X \setminus \Sigma) \to L^{i+1}(X \setminus \Sigma))}{\operatorname{Im}(L^{i-1}(X \setminus \Sigma) \to L^i(X \setminus \Sigma))}$

Conjecture $H^*_{(2)}(X)$ is isomorphic to intersection homology

What about ring structure ?

② Simplicial homology

Definition σ is an n-simplex in $\mathbb{R}^n$ if it is the convex hull of points $v_0, \dots, v_n \in \mathbb{R}^n$ affinely independant.
orientation ordering of the vertices up to even permutation

Definition a Simplicial complex in $\mathbb{R}^n$ is a set N of simplices s.t.
(i) If $\sigma \in N$ then any face of σ is in N.
(ii) If $\sigma, \tau \in N$ and $\sigma \cap \tau \neq \phi$, then $\sigma \cap \tau$ is a simplex whose vertices belong to σ and τ
(iii) $\forall\, x \in \sigma \in N$, there is a neighbourhood U of x s.t. $U \cap \tau = \phi$ except for a finite number of $\tau \in N$.

Support : $|N| = \bigcup_{\sigma \in N} \sigma$, $N^{(i)} \subset N$: dim i

Triangulation of top. space X = homeomorphism $T : |N| \to X$.

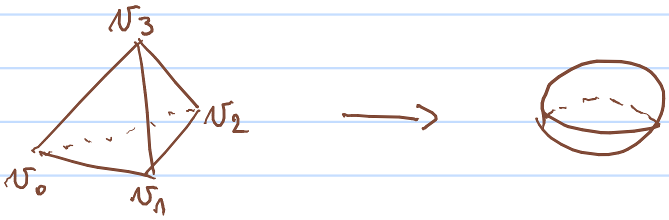
$\leadsto$ $C_i(N) = \bigoplus_{\sigma \in N^{(i)}} \mathbb{C} \cdot \sigma \big/ (\bar{\sigma} = -\sigma)$ — oriented simplex; $\bar{\sigma}$: simplex σ with the reverse orientation

$$\partial : C_i(N) \to C_{i-1}(N)$$
$$\sigma_{v_0 \cdots v_n} \mapsto \sum (-1)^j \, \tau_{v_0 \cdots \hat{v_j} \cdots v_n}$$

Definition $$H_i^T(N) = \frac{\ker\left(C_i(N) \to C_{i-1}(N)\right)}{\operatorname{Im}\left(C_{i+1}(N) \to C_i(N)\right)}$$

Examples

a) Sphere



$\operatorname{Im}\left(C_1(N) \to C_0(N)\right) = \langle v_i - v_j \rangle$ codimension 1 $\to$ dim 3

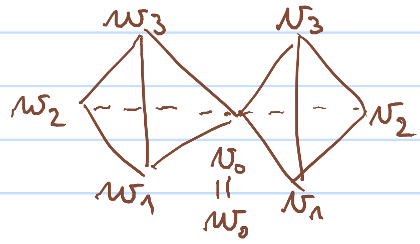
So $\ker\left(C_1(N) \to C_0(N)\right)$ dim $6 - 3 = 3$

$\ker\left(C_2(N) \to C_1(N)\right)$ dim 1 generated by ∂(3-dim simplex)

So $\operatorname{Im}\left(C_2(N) \to C_1(N)\right)$ dim $4 - 1 = 3$

thus : $H_2^T(N) = \mathbb{C}$, $H_1^T(N) = 0$, $H_0^T(N) = \mathbb{C}$

b) Singular conic

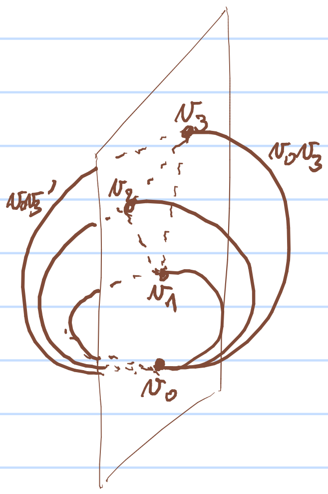


$\mathrm{Im}\,(C_1 \to C_0)$ codimension 1, dim 6
$\ker\,(C_1 \to C_0)$ dim $12 - 6 = 6$

$\ker\,(C_2 \to C_1)$ generated by ∂(the two 3-simplices) dim 2
$\mathrm{Im}\,(C_2 \to C_1)$ dim $8 - 2 = 6$

$H_0^T(N) = \mathbb{C}$ $\quad H_1^T(N) = 0$ $\quad H_2^T(N) = \mathbb{C}^2$

c) Singular elliptic curve



$\mathrm{Im}\,(C_1 \to C_0)$ dim 3
$\ker\,(C_1 \to C_0)$ dim $9 - 3 = 6$

$\ker\,(C_2 \to C_1)$ dim 1 generated by the sum of all 2-simplices, which is ∂(the whole 3-space)
$\mathrm{Im}\,(C_2 \to C_1)$ dim $6 - 1 = 5$

$H_0^T(N) = \mathbb{C}$ $\quad H_1^T(N) = \mathbb{C}$ $\quad H_2^T(N) = \mathbb{C}$

Definition $\tilde{T} : |\tilde{N}| \to X$ is a refinement of $T : |N| \to X$ if $\forall\, \tilde{\sigma} \in \tilde{T}$ $\exists\, \sigma \in T$: $\tilde{T}(\tilde{\sigma}) \subset T(\sigma)$

$\to$ get a map $C_i(N) \to C_i(\tilde{N})$

Definition This defines a complex $(C_i(X), \partial) = \varinjlim_{T} (C_i^T(X), \partial^T)$

element of this limit is $\sigma \in C_i^T(X)$, with $\sigma \sim \sigma'$ if $\exists\, \tilde{T}$ refinement of T and T' s.t. σ and σ' are equal in $C_i^{\tilde{T}}(X)$.

Definition $$H_i^{simp}(X) = \frac{\ker(C_i(X) \to C_{i-1}(X))}{\mathrm{Im}\,(C_{i+1}(X) \to C_i(X))}$$

Theorem (2.2.3) If $T : |N| \to X$ is any triangulation of a topological space X, then

$$H_*^T(X) = H_*^{simp}(X) = H_*^{sing}(X)$$

singular homology, defined with continuous simplices with values in X.

Homology with closed support (Borel-Moore homology)

Definition $C_i^T((X)) = \{ \sum_{\sigma \in N^{(i)}} d_\sigma \sigma \}$ (infinite sum)

$C_i((X)) = \varinjlim_T C_i^T((X))$.

Observe: the support of $\sum d_\sigma \sigma = \bigcup_{\sigma : d_\sigma \neq 0} T(\sigma) \subset X$ is closed, since any simplicial complex is locally finite.

Definition $H_i^{cl}(X) = \dfrac{\ker(C_i((X)) \to C_{i-1}((X)))}{\operatorname{Im}(C_{i+1}((X)) \to C_i((X)))}$.

③ Whitney stratification

Definition (Whitney stratification)
A Whitney stratification is a filtration $X = X_n \supset X_{n-1} \supset \cdots \supset X_0$ of X by closed subvarieties s.t. $\forall j$, $X_j \setminus X_{j-1}$ is empty or smooth, and such that the connected components S_α of the $X_j \setminus X_{j-1}$ satisfy:

(a) If $x_k \in S_\alpha$, $\lim_{k\to\infty} x_k = y \in S_\beta$, $\lim_{k\to\infty} T_{x_k} S_\alpha$ exists, Then $T_y S_\beta \subset \lim_{k\to\infty} T_{x_k} S_\alpha$

(b) If $x_k \in S_\alpha$, $y_k \in S_\beta$ are such $\lim_{k\to\infty} x_k = \lim_{k\to\infty} y_k = y \in S_\beta$ $\lim_{k\to\infty} \langle x_k y_k \rangle$ and $\lim_{k\to\infty} T_{x_k} S_\alpha$ exist, then
$\lim_{k\to\infty} \langle x_k, y_k \rangle \subset \lim_{k\to\infty} T_{x_k} S_\alpha$.

Example $X = \{ (x,y,z) \in \mathbb{C}^3 \mid x^4 + y^4 = xyz \}$

$X_2 = X \quad X_1 = \{(0,0,z)\} \quad X_0 = \emptyset$

$df = (4x^3 + yz)\,dx + (4y^3 + xz)\,dy + xy\,dz$

$df = 0 \Rightarrow xy = 0$

Supp $x = 0$. Alors $4y^3 + 0 = 0$ $y = 0$.

$df = 0 \Leftrightarrow x = y = 0$

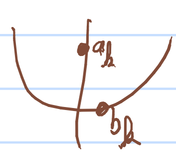
Condition (b) is not satisfied : $S_\alpha = X \setminus X_1$ $S_\beta = X_1$

$a_k = \left(\frac{1}{k^2}, \frac{1}{k^2}, \frac{2}{k^4} \right) \in S_\alpha \qquad b_k = (0, 0, \frac{1}{k})$

$\lim_{k\to\infty} \langle a_k b_k \rangle = X_1$

$\lim_{k\to\infty} T_{a_k} S_\alpha = \{ z = 0 \}$

$X \cap (x = y) : 2x^4 = x^2 z$
$\Leftrightarrow x = 0$ ou $z = 2x^2$



We would have a Whitney stratification setting $X_0 = \{(0,0,0)\}$.

Consequence of Whitney conditions $\forall x, y \in S_\alpha$, $\exists f$ homeomorphism of X s.t. $f(x) = y$ and $f(X_j) = X_j$.

Theorem (Whitney 65) Any quasi-projective variety of pure dim n has a Whitney stratification.

Theorem (Łojasiewicz 64, Goresky 78) Let $X = X_n \supset X_{n-1} \supset \cdots \supset X_0$ be a Whitney stratification of a complex quasi-projective variety X of pure dim n. Then there is a triangulation of X compatible with the stratification

ie each X_i is union of simplices.

4 Intersection homology

Definition A perversity is a finite sequence $p = (p_2, \dots, p_{2n})$ of integers such that $p_2 = 0$ and $p_{k+1} \in \{p_k, p_k + 1\}$.

Example $\bar{0} = (0, \dots, 0)$

$\bar{t} = (0, 1, \dots, 2n-2)$ $\quad \bar{m} = (0, 0, 1, 1, 2, 2, \dots, n-1)$

complementary perversity of $\bar{p}$ is $\bar{t} - \bar{p} = (0, 1-p_3, \dots, 2n-2-p_{2n-2})$.

Definition $IC_i^{\bar{p}, T}(X) \subset C_i^T(X)$: set of i-chains η s.t.

$$\forall k \geq 1 \begin{cases} \dim_{\mathbb{R}} |\eta| \cap X_{n-k} \leq i - 2k + p_{2k} & i-k-1 \\ \dim_{\mathbb{R}} |\partial\eta| \cap X_{n-k} \leq i - 2k + p_{2k} - 1 & i-k-2 \end{cases}$$

(middle perversity)

Definition

$$IC_i^{\bar{p}}(X) = \varinjlim_T IC_i^{\bar{p}, T}(X)$$

$$IH_i^{\bar{p}}(X) = \frac{\ker(IC_i^{\bar{p}}(X) \to IC_{i-1}^{\bar{p}}(X))}{\mathrm{Im}(IC_{i+1}^{\bar{p}}(X) \to IC_i^{\bar{p}}(X))}$$

$$IH_i(X) = IH_i^{\bar{m}}(X)$$

Remark Can also define intersection homology with closed support.

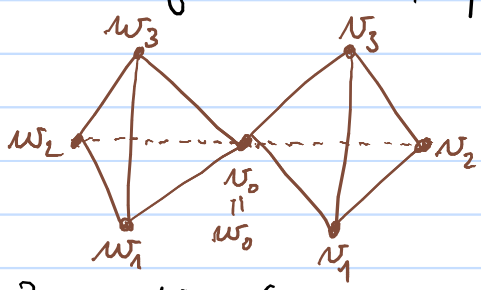
⑤ Examples

If $\dim X = 1$,
the condition on IC_i reads $(k=1)$

$$\dim_{\mathbb R} |\eta| \cap X_0 \le i-2$$
$$\dim_{\mathbb R} |\partial \eta| \cap X_0 \le i-3$$

ie: if $i \le 1$, $|\eta| \cap X_0 = \emptyset$
if $i = 2$, $|\partial\eta| \cap X_0 = \emptyset$

a) Union of two spheres



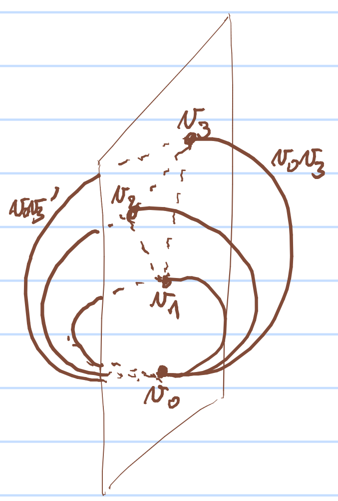
$IC_0 = \mathbb C\{v_1, v_2, v_3, w_1, w_2, w_3\}$ dim 6
$IC_1 = \mathbb C\{v_1v_2, v_1v_3, v_2v_3, w_1w_2, w_1w_3, w_2w_3\}$ dim 6
$IC_2 = \ker(C_2 \xrightarrow{\partial} C_1) \oplus \mathbb C\{v_1v_2v_3, w_1w_2w_3\}$ dim 4

$\mathrm{Im}(IC_1 \to IC_2) = \{\sum_{i=1}^3 \lambda_i v_i + \sum_{i=1}^3 \mu_i w_i \mid \Sigma\lambda_i = \Sigma\mu_i = 0\}$ dim 4
$\ker(IC_1 \to IC_2)$ dim $6-4=2$

$\ker(IC_2 \to IC_1)$ dim 2
$\mathrm{Im}(IC_2 \to IC_1)$ dim 2

$\Rightarrow IH_0(X) = \mathbb C^2$
$IH_1(X) = 0$
$IH_2(X) = \mathbb C^2$

b) Singular elliptic curve



$IC_0 = \mathbb C\{v_1, v_2, v_3\}$ dim 3
$IC_1 = \mathbb C\{v_1v_2, v_2v_3, v_3v_1\}$ dim 3
$IC_2 = \mathbb C\cdot(v_0v_1v_2 + v_0v_2v_3 + v_0v_3v_1) \oplus \mathbb C(v_0v_1v_2' + v_0v_2v_3' + v_0v_3v_1')$

$\partial(\;—\;) = v_1v_2 + v_2v_3 + v_3v_1$

$\mathrm{Im}(IC_1 \to IC_0)$ dim 2
$\ker(IC_1 \to IC_0)$ dim $3-2=1$

$\ker(IC_2 \to IC_1)$ dim 1
$\mathrm{Im}(IC_2 \to IC_1)$ dim 1

$\Rightarrow IC_0(X) = \mathbb C$
$IC_1(X) = 0$
$IC_2(X) = \mathbb C$

⑥ Properties of intersection homology

Isolated singularities

If X is smooth, then $X_n = X \supset X_{n-1} = \emptyset \supset \cdots \supset X_0 = \emptyset$ is a Whitney filtration, and $IC_i(X) = C_i(X)$.

So: $IH_i^{\bar{p}}(X) = H_i(X)$ for any perversity $\bar{p}$.

Proposition Let X be a quasi-projective variety with one isolated singularity $x \in X$. Then

$$IH_i(X) = \begin{cases} H_i(X) & \text{if } i > n \\ \operatorname{Im}\left(H_n(X - \{x\}) \to H_n(X)\right) & \text{if } i = n \\ H_i(X - \{x\}) & \text{if } i < n \end{cases}$$

Pf $X_n = X \supset X_{n-1} = \{x\} \supset \cdots \supset X_0 = \{x\}$ is a Whitney stratification.

We have $IC_i(X) = \left\{ \eta \in C_i(X) \;\middle|\; \begin{array}{l} \dim |\eta| \cap \{x\} \leq i - n - 1 \\ \text{and } \dim |d\eta| \cap \{x\} \leq i - n - 2 \end{array} \right\}$

Thus we have:

for $i \leq n$, $IC_i(X) = C_i(X \setminus \{x\})$ $\rightsquigarrow$ for $i \leq n-1$, $IH_i(X) = H_i(X \setminus \{x\})$

for $i \geq n+2$, $IC_i(X) = C_i(X)$ $\rightsquigarrow$ for $i \geq n+2$, $IH_i(X) = H_i(X)$

and: $IC_{n+1}(X) = \{\eta \mid |d\eta| \not\ni x\}$

Thus $\ker\left(IC_{n+1}(X) \xrightarrow{\partial} IC_n(X)\right) = \ker\left(C_{n+1}(X) \xrightarrow{\partial} C_n(X)\right)$

so $IH_{n+1}(X) = H_{n+1}(X)$.

and $\operatorname{Im}\left(IC_{n+1}(X) \xrightarrow{\partial} IC_n(X)\right) = \operatorname{Im}\left(C_{n+1}(X) \xrightarrow{\partial} C_n(X)\right) \cap IC_n(X)$

so $IH_n(X) = \operatorname{Im}\left(H_n(X \setminus \{x\}) \to H_n(X)\right)$. ∎

Normal varieties

def A domain A is *integrally closed* if $\forall\, a, b \in A$, if $\frac{a}{b} \in \operatorname{Frac}(A)$ is a root of an integral polynomial with coefficients in A, then $\frac{a}{b} \in A$.

def An integral scheme is *normal* if all its local rings are integrally closed.

Normalisation Let X be an integral scheme. There is a scheme $\tilde{X}$ and a morphism $\tilde{X} \to X$ with the universal property:

$$\begin{tikzcd} \tilde{X} \arrow[d] & \\ X & Z \arrow[l, "\text{dominant}"] \arrow[ul, dashed, "\exists!"'] \end{tikzcd}$$

normal

Such $\tilde{X}$ is called the *normalisation* of X, and $\tilde{X} \to X$ is finite provided X has finite type over a field.

Proposition [Goresky - MacPherson]
Let X be a normal quasi-projective variety of dimension n. Then

$$IH_i^{\bar{t}}(X) \simeq H_i(X)$$
$$IH_i^{\bar{0}}(X) \simeq H^{2n-i}(X)$$

If $\pi: \tilde{X} \to X$ is the normalisation of X, then
$$IH_i^{\bar{p}}(\tilde{X}) \simeq IH_i^{\bar{p}}(X) \quad \forall \bar{p}.$$

Example For the singular conic, $IH_*(X) = \mathbb{C}^2 \oplus 0 \oplus \mathbb{C}^2$

cubic, $IH_*(X) = \mathbb{C} \oplus 0 \oplus \mathbb{C}$

Poincaré duality

Proposition Let X be a projective variety of pure dimension n. If $\bar{p}$ and $\bar{q}$ are complementary perversities, then there is a non-degenerate pairing
$$IH_i^{\bar{p}}(X) \otimes IH_j^{\bar{q}}(X) \to \mathbb{C} \quad \text{when } i+j = 2n$$

For $\bar{p} = \bar{q} = \bar{m}$: $IH_i(X) \otimes IH_j(X) \to \mathbb{C}$ when $i+j = 2n$.

Remark There is no natural ring structure on $IH^*(X)$. However, if X is normal and projective, we have
$$IH^i(X) \otimes IH^j(X) \to H_{2n-i-j}(X).$$